Instability of Spinning Space Stations Due to Crew Motion

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The stability of a spinning space station due to periodic motions of the crew is considered. It is shown by a few examples that instability may occur if the period of an astronaut's motion bears certain ratios to the half-period of the spin of the satellite. For example, if he moves back and forth along a radius of a circular, planar satellite, instability will occur when the period of his motion is approximately an integral multiple of the half-period of the satellite spin. A similar conclusion holds if the astronauts move with constant speed or oscillate periodically in circumferential direction. The heavier the moving masses are, relative to the satellite, or the larger their amplitude of motion, the wider will be the regions of instability. General equations are given which are useful for analyzing other modes of crew motion or for other problems such as passive damping of a spinning satellite.

Nomenclature

a	= parameter defined by Eq. (28)
$\mathbf{a}_0$	= acceleration of origin O
A, B, and C	= moments of inertia about x, y, and z axes, respectively
$[A]$, $[S]$, $[U]$	= square matrices defined in text
$\{F\}$, $\{Q\}$, $\{V\}$	= column matrices defined in text
$\mathbf{H}_0(H_x, H_y, H_z)$	= moment of momentum with respect to origin O
$\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$	= unit vectors in x, y, and z directions
$[I]$	= inertia matrix
$[T(\omega)]$	= square matrix defined in text
I_x, I_y, and I_z	= moment of inertia about x, y, and z axes, respectively
I_{xy}, I_{yz}, and I_{zx}	= products of inertia
m	= mass of moving body
$\mathbf{v}_R$	= velocity of m relative to the x, y, z coordinates
M	= mass of satellite (excluding crew)
$\mathbf{M}_0(M_x, M_y, M_z)$	= moment of external forces about origin O
p	= circular frequency of crew motion
q	= parameter defined by Eq. (28)
T	= period of motion of crew
u	= variable defined by Eq. (26)
x, y, and z	= moving frame of reference
β	= parameter defined by Eq. (21)
ϵ	= ratio $2m\rho^2/A$
θ	= angular position of moving body (Fig. 2)
θ_0	= amplitude of angular oscillatory motion
θ_0, θ_{1c}, and θ_{1s}, ...	= parameters defined by Eq. (43)
μ	= characteristic exponent of solution of Mathieu's equation
$\boldsymbol{\rho}$	= relative positive vector with respect to moving origin O
$\boldsymbol{\omega}(\omega_x, \omega_y, \omega_z)$	= angular velocity of moving frame of reference
$\boldsymbol{\omega}'(\omega_x', \omega_y', \omega_z')$	= perturbations defined by Eq. (6)
ω_s	= spin speed of satellite
ω_{s0}	= speed of spin before crew motion
Ω	= predominant angular speed [see Eq. (6)]
τ	= nondimensional time
η	= parameter defined by Eq. (27)

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Introduction

We shall consider the question of the motion of astronauts in a space station in free flight. When the astronauts move, will the station tumble? Can instability be avoided?

Since action and reaction between the satellite and the astronauts consist of internal forces, the free-flight trajectory of the center of mass will not be affected by the motion of man. If we consider motions that occur in a time interval during which the arc length traversed by the satellite is small compared with the radius of curvature of the trajectory, we may regard the satellite as in a free rectilinear motion at constant velocity. We shall consider the problem under this approximation.

Equations of Motion

Let x, y, and z be the moving frame of reference with the origin O located at a convenient point in the structure as shown in Fig. 1. Then $\boldsymbol{\omega}(\omega_x, \omega_y, \omega_z)$ is the angular velocity of the moving frame of reference, and the moment of momentum with respect to O,

$$\mathbf{H}_0 = \int \boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})dm + \int \boldsymbol{\rho} \times \mathbf{v}_R dm \tag{1}$$

has the following components:

$$\left.\begin{aligned} H_x &= I_x\omega_x - I_{xy}\omega_y - I_{xz}\omega_z + \mathbf{i}\cdot\int\boldsymbol{\rho}\times\mathbf{v}_R dm \\ H_y &= -I_{xy}\omega_x + I_y\omega_y - I_{yz}\omega_z + \mathbf{j}\cdot\int\boldsymbol{\rho}\times\mathbf{v}_R dm \\ H_z &= -I_{xz}\omega_x - I_{yz}\omega_y + I_z\omega_z + \mathbf{k}\cdot\int\boldsymbol{\rho}\times\mathbf{v}_R dm \end{aligned}\right\} \tag{2}$$

The moment equation

$$\mathbf{M}_0 = (d\mathbf{H}/dt) + \boldsymbol{\omega} \times \mathbf{H} - \mathbf{a}_0 \times \int \boldsymbol{\rho} dm \tag{3}$$

can then be resolved into the x, y, and z components

$$\left.\begin{aligned} M_x &= \dot{H}_x - H_y\omega_z + H_z\omega_y - \mathbf{i}\cdot(\mathbf{a}_0 \times \int\boldsymbol{\rho} dm) \\ M_y &= \dot{H}_y - H_z\omega_x + H_x\omega_z - \mathbf{j}\cdot(\mathbf{a}_0 \times \int\boldsymbol{\rho} dm) \\ M_z &= \dot{H}_z - H_x\omega_y + H_y\omega_x - \mathbf{k}\cdot(\mathbf{a}_0 \times \int\boldsymbol{\rho} dm) \end{aligned}\right\} \tag{4}$$

where the last column of terms can be dropped in the case where the origin coincides with the center of mass of the system.

On substituting (2) into (4) and reducing, we obtain the following nonlinear equation of motion in case the origin is located at the center of mass:

$$[I]\{\dot{\omega}\} + [\dot{I}]\{\omega\} + [T(\omega)]\{\omega\} + \{Q\} = 0 \tag{5}$$

where

$$[I] = \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z \end{bmatrix}$$

$$[T(\omega)] = \begin{bmatrix} 0 & (-I_{zx}\omega_x - I_{zy}\omega_y + I_z\omega_z) & -(-I_{yx}\omega_x + I_y\omega_y - I_{yz}\omega_z) \\ -(-I_{zx}\omega_x - I_{zy}\omega_y + I_z\omega_z) & 0 & (I_x\omega_x - I_{xy}\omega_y - I_{xz}\omega_z) \\ (-I_{yx}\omega_x + I_y\omega_y - I_{yz}\omega_z) & -(I_x\omega_x - I_{xy}\omega_y - I_{xz}\omega_z) & 0 \end{bmatrix}$$

$$\{Q\} = \begin{Bmatrix} \int(y\ddot{z} - z\ddot{y})dm + \omega_y\int(x\dot{y} - y\dot{x})dm - \omega_z\int(z\dot{x} - x\dot{z})dm \\ \int(z\ddot{x} - x\ddot{z})dm + \omega_z\int(y\dot{z} - z\dot{y})dm - \omega_x\int(x\dot{y} - y\dot{x})dm \\ \int(x\ddot{y} - y\ddot{x})dm + \omega_x\int(z\dot{x} - x\dot{z})dm - \omega_y\int(y\dot{z} - z\dot{y})dm \end{Bmatrix}$$

The present problem is to study the motion of the body when the masses move in a specific manner. In particular, we would like to discover all undesirable types of motion of the masses, so as to avoid them in practice.

Linearization

In the following, we shall assume that the origin of coordinates is centroidal and that ω_z predominates. Let

$$\left.\begin{aligned} &\omega_z = \Omega + \omega_z' \\ &\omega_x = \omega_x' \qquad \omega_y = \omega_y' \\ &\Omega \gg (\omega_x', \omega_y', \omega_z') \end{aligned}\right\} \tag{6}$$

Then, neglecting terms of order ω'^2, we can obtain the following linear equations with variable coefficients:

$$\{\dot{\omega}'\} + [A(t)]\{\omega'\} = \{F(t)\} \tag{7}$$

where

$$[A] = [I]^{-1}\{[\dot{I}] + \Omega[S] + [U]\}$$

$$\{F\} = -[I]^{-1}\{V\}$$

$$[I] = \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z \end{bmatrix}$$

$$[S] = \begin{bmatrix} I_{yx} & (I_z - I_y) & 2I_{yz} \\ -(I_z - I_x) & -I_{xy} & -2I_{xz} \\ -I_{yz} & I_{xz} & 0 \end{bmatrix}$$

$$[U] = \begin{bmatrix} 0 & \int(x\dot{y} - y\dot{x})dm & \int(x\dot{z} - z\dot{x})dm \\ \int(y\dot{x} - x\dot{y})dm & 0 & \int(y\dot{z} - z\dot{y})dm \\ \int(z\dot{x} - x\dot{z})dm & \int(z\dot{y} - y\dot{z})dm & 0 \end{bmatrix}$$

$$\{V\} = \begin{Bmatrix} -I_{xz}\dot{\Omega} - \Omega\dot{I}_{xz} + \Omega^2 I_{yz} + \int(y\ddot{z} - z\ddot{y})dm + \Omega\int(x\dot{z} - z\dot{x})dm \\ -I_{yz}\dot{\Omega} - \Omega\dot{I}_{yz} - \Omega^2 I_{zx} + \int(z\ddot{x} - x\ddot{z})dm + \Omega\int(y\dot{z} - z\dot{y})dm \\ I_z\Omega + \Omega\dot{I}_z + 0 + \int(x\ddot{y} - y\ddot{x})dm + 0 \end{Bmatrix}$$

It is clear that the matrix $[A(t)]$ is, in general, nonsymmetric. In practice, one should choose the coordinates such that the inertia matrix $[I]$ can be diagonalized or be as much a constant matrix as possible.

Stability

Equation (7), which is linear and is valid when $|\omega'| \ll \Omega$, can be used to study the trend of motion under small perturbations. If, following a small initial disturbance, the solution ω' tends toward zero or remains bounded as $t \to +\infty$, we say that the solution is stable; if ω' tends toward $\pm\infty$ as $t \to +\infty$, it is said to be unstable. The instability so defined must be interpreted as an initial tendency. In reality, as $|\omega'|$ grows to be comparable with Ω, the linear Eq. (7) must give way to the nonlinear Eq. (5). Nevertheless the initial tendency toward instability may warrant attention, because although instability as defined previously does not necessarily mean disaster, it is probably wise to be aware of it.

It should be noted that a satellite system with astronauts on board is a nonconservative system. The astronauts are capable of converting chemical energy into mechanical energy or of dissipating mechanical energy into heat. Hence, when instability occurs, the energy required may be supplied by the astronauts. On the other hand, in a multidegree-of-freedom system, there is a possibility that instability of one mode of oscillation may occur by drawing energy from another mode of motion. For example, tumbling instability of a satellite may derive its energy from the spinning motion. The situation is similar to the possible generation of flutter of the wings of an airplane at the expense of its forward speed of flight.

Circumferential Motion of Crew

In the first special case, we investigate the circumferential motion of two crew members on diametrically opposite sides of a symmetrical space station as shown in Fig. 2. If the x and y axes are fixed in the space station, the moments and products of inertia about the x, y, and z axes would be a function of θ or time, and Eq. (4) would lead to time-varying coefficients. To avoid this undesirable complication, the x and y axes may be placed through the center of mass, as shown in Fig. 2b, with the x axis passing through the crew members designated by m. With this choice of axes, the products of inertia are zero, and the moments of inertia will be constant.

Letting the moments of inertia of the space station (without the two crew members $2m$) be $I_x = I_y = A$ and $I_z = C$, we will consider the case where the crew moves circumferentially with respect to the station with an angular speed $\dot{\theta}$ The moments of momentum, from Eq. (2), are

$$\begin{aligned} &H_x = A\omega_x \qquad H_y = (A + 2m\rho^2)\omega_y \\ &H_z = (C + 2m\rho^2)\omega_z - C\dot{\theta} \end{aligned} \tag{8}$$

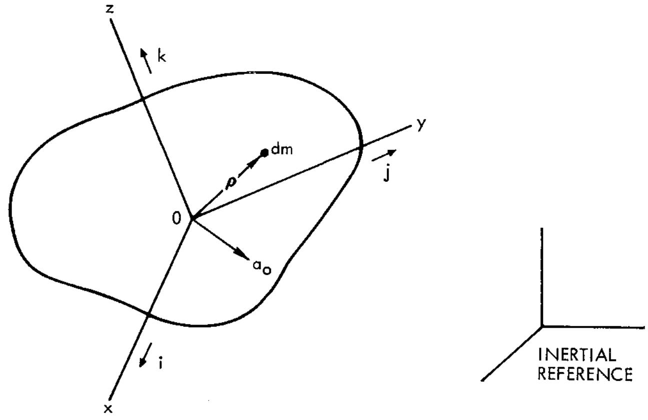

Fig. 1 Moving axes x, y, and z with arbitrary origin O.

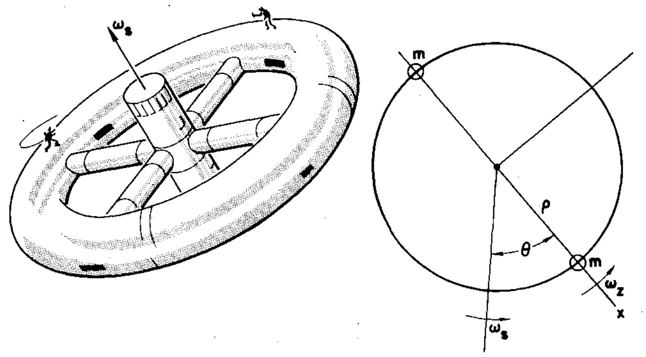

Fig. 2 Space station and coordinates for circumferential motion of crew.

and Eq. (4) reduces to

$$\left.\begin{aligned} &A\dot{\omega}_x - (A + 2m\rho^2)\omega_y\omega_z + [(C + 2m\rho^2)\omega_z - C\dot{\theta}]\omega_y = 0 \\ &(A + 2m\rho^2)\dot{\omega}_y - [(C + 2m\rho^2)\omega_z - C\dot{\theta}]\omega_x + A\omega_x\omega_z = 0 \\ &(d/dt)[(C + 2m\rho^2)\omega_z - C\dot{\theta}] - A\omega_x\omega_y + (A + 2m\rho^2)\omega_x\omega_y = 0 \end{aligned}\right\} \quad (9)$$

To linearize these equations, we will assume that ω_z predominates and make the substitution

$$\begin{aligned} &\omega_x = \omega_x' \qquad \omega_y = \omega_y' \\ &\omega_z = \Omega + \omega_z' = \omega_s + \dot{\theta} + \omega_z' \end{aligned} \quad (10)$$

where the primed quantities are small in comparison with Ω. We choose $\Omega = \omega_s + \dot{\theta}$, where ω_s is the spin speed of the satellite about the z axis. Neglecting quantities of order $0(\omega'^2)$, the linearized equations become

$$\dot{\omega}_x' + \{[(C/A) - 1]\omega_s - \dot{\theta}\}\omega_y' = 0 \quad (11)$$

$$\dot{\omega}_y' - \frac{1-\epsilon}{1+\epsilon}\left[\left(\frac{C}{A}\frac{1}{1-\epsilon} - 1\right)\omega_s - \dot{\theta}\right]\omega_x' = 0 \quad (12)$$

$$\dot{\omega}_z' = -1/[1 + \epsilon(A/C)]\ddot{\theta} \quad (13)$$

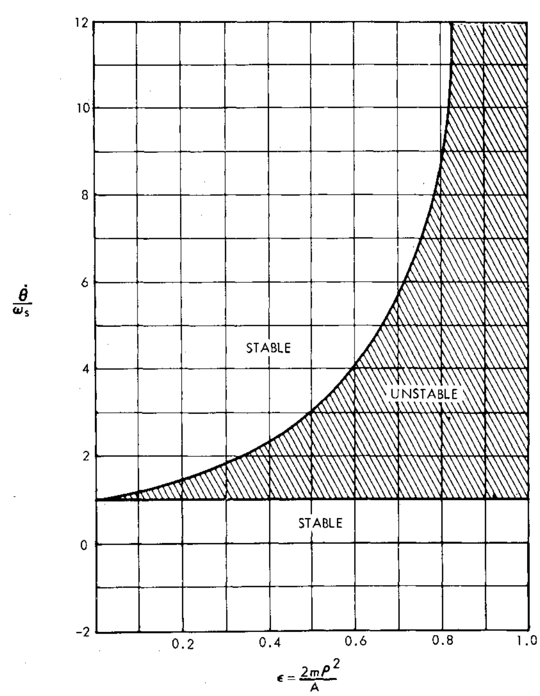

Fig. 3 Instability under circumferential motion of crew.

where $\epsilon = 2m\rho^2/A$ is the ratio of the polar moment of inertia of the astronauts to that of the satellite about the y axis. Equation (13) gives

$$\omega_z' = -1/[1 + \epsilon(A/C)]\dot{\theta} + \text{const} \quad (14)$$

Equations (11) and (12) can be combined by differentiating (12) with respect to time and then substituting $\dot{\omega}_x'$ and ω_x' from Eqs. (11) and (12), respectively. The result is

$$\begin{aligned} &\ddot{\omega}_y' - \left[\left(\frac{C}{A}\frac{1}{1-\epsilon} - 1\right)\dot{\omega}_s - \ddot{\theta}\right] \times \\ &\left[\left(\frac{C}{A}\frac{1}{1-\epsilon} - 1\right)\omega_s - \dot{\theta}\right]^{-1}\dot{\omega}_y' + \\ &\frac{1-\epsilon}{1+\epsilon}\left[\left(\frac{C}{A}\frac{1}{1-\epsilon} - 1\right)\omega_s - \dot{\theta}\right] \times \\ &\left[\left(\frac{C}{A} - 1\right)\omega_s - \dot{\theta}\right]\omega_y' = 0 \end{aligned} \quad (15)$$

Let us consider some special cases.

1) The crew members move around at a constant speed ($\dot{\theta}$ = const and ω_s = const). Then Eq. (15) becomes

$$\ddot{\omega}_y' + \lambda^2\omega_y' = 0 \quad (16)$$

where

$$\lambda^2 = \frac{1-\epsilon}{1+\epsilon}\left[\left(\frac{C}{A}\frac{1}{1-\epsilon} - 1\right)\omega_s - \dot{\theta}\right] \times \left[\left(\frac{C}{A} - 1\right)\omega_s - \dot{\theta}\right] \quad (17)$$

The space station becomes unstable when $\lambda^2 < 0$, i.e., for $\dot{\theta}/\omega_s$ in the interval

$$(C/A) - 1 < \dot{\theta}/\omega_s < (C/A)[1/(1 - \epsilon)] - 1 \quad (18)$$

Figure 3 is a plot of the stable and unstable regions as found from Eq. (18) for the special case $C = 2A$. It indicates that, for $\dot{\theta}/\omega_s < 1.0$, the system is stable for all values of ϵ. For a given value of ϵ, the system becomes unstable for a range of speed ratio $\dot{\theta}/\omega_s$ and then becomes stable again.

Figure 4 shows the same result in terms of the initial spin speed ω_{s0} of the station before the crew motion was initiated. The relationship here is established from the conservation of moment of momentum

$$(C + 2m\rho^2)\omega_{s0} = C\omega_s + 2m\rho^2(\omega_s + \dot{\theta})$$

from which

$$\frac{\dot{\theta}}{\omega_{s0}} = \frac{1}{(\omega_s/\dot{\theta}) + [\epsilon/(2 + \epsilon)]} \quad (19)$$

Thus from the points $\dot{\theta}/\omega_s$ and ϵ of Fig. 3, the stability boundaries can be translated in terms of $\dot{\theta}/\omega_{s0}$ of Fig. 4.

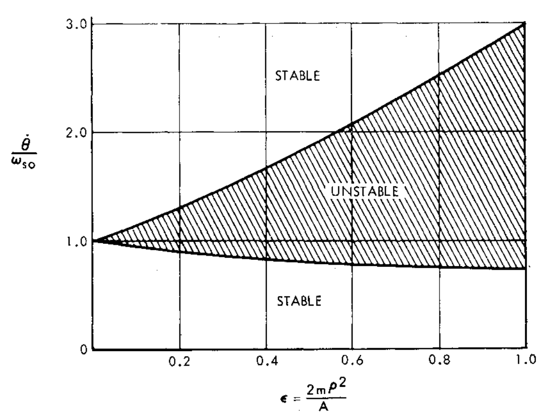

Fig. 4 Instability in terms of initial angular rate ω_{s0}.

It is of interest to consider further the unsymmetric case of a single mass m in circumferential motion. If the previous origin at the center of the space station is maintained, the center of mass will lie on the x axis passing through m, and the origin O will move in a circle around the center of mass, which will remain stationary. Again the term $\mathbf{a}_0 \times \int \boldsymbol{\varrho}\, dm$ in Eq. (3) is zero because $\mathbf{a}_0$ is now parallel to $\int \boldsymbol{\varrho}\, dm$, and Eqs. (11–13) will differ only in the term $2m$, which now is replaced by m. We find for this case that the same stability condition indicated by Eq. (18) holds if ϵ is now defined as $m\rho^2/A$. Thus the results of Figs. 3 and 4 apply to the unsymmetric case with $2m$ replaced by m.

If, next, the circumferential motion of the two astronauts takes place in a plane at a distance z from the previous center of mass plane, the equation of motion corresponding to Eq. (16) now becomes

$$\ddot{\omega}_y' + \omega_s^2 \left(\frac{1-\beta}{1+\beta} - \frac{\dot{\theta}}{\omega_s}\right) \times \left[\frac{(1+\epsilon-\beta) - (1-\epsilon+\beta)\dot{\theta}/\omega_s}{1+\epsilon+\beta}\right] \omega_y' = 0 \quad (20)$$

where

$$\epsilon = 2m\rho^2/A \qquad \beta = [2mM/(M+2m)](z^2/A) \quad (21)$$

The preceding equation indicates that the lower bound of instability is lowered from $\dot{\theta}/\omega_s = 1$ to $\dot{\theta}/\omega_s = (1-\beta)/(1+\beta)$ and that $(\epsilon - \beta)$ must now replace ϵ of the previous examples. The stable and unstable regions are then shown in Fig. 5.

2) The crew members are undergoing harmonic oscillations [$\theta(t) = \theta_0 \sin pt$ and $\omega_s = $ const]. In this case

$$\dot{\omega}_x' + \{[(C/A) - 1]\omega_s - \theta_0 p \cos pt\}\omega_y' = 0 \quad (22)$$

$$\dot{\omega}_y' - \frac{1-\epsilon}{1+\epsilon}\left[\left(\frac{C}{A}\frac{1}{1-\epsilon} - 1\right)\omega_s - \theta_0 p \cos pt\right]\omega_x' = 0 \quad (23)$$

or

$$\ddot{\omega}_y' - \frac{\theta_0 p^2 \sin pt}{\left(\frac{C}{A}\frac{1}{1-\epsilon} - 1\right)\omega_s - \theta_0 p \cos pt}\,\dot{\omega}_y' + \frac{1-\epsilon}{1+\epsilon}\left[\left(\frac{C}{A}\frac{1}{1-\epsilon} - 1\right)\omega_s - \theta_0 p \cos pt\right] \times \left[\left(\frac{C}{A} - 1\right)\omega_s - \theta_0 p \cos pt\right]\omega_y' = 0 \quad (24)$$

If θ_0 is very small in comparison with unity, Eq. (24) may be linearized with respect to θ_0 to obtain

$$\ddot{\omega}_y' - \frac{1}{\{(C/A)[1/(1-\epsilon)] - 1\}\omega_s}\theta_0 p^2 \sin pt\, \dot{\omega}_y' + \frac{1-\epsilon}{1+\epsilon}\left[\left(\frac{C}{A}\frac{1}{1-\epsilon} - 1\right)\left(\frac{C}{A} - 1\right)\omega_s^2 - \left(\frac{C}{A}\frac{2-\epsilon}{1-\epsilon} - 2\right)\omega_s\theta_0 p \cos pt\right]\omega_y' = 0 \quad (25)$$

Let

$$\omega_y' = \exp(-\eta \cos pt)u \quad (26)$$

where

$$\eta = \theta_0 p/2\{(C/A)[1/(1-\epsilon) - 1]\omega_s\} \quad (27)$$

$$\left.\begin{aligned} pt &= 2\tau + \pi \\ a &= 4\frac{1-\epsilon}{1+\epsilon}\left(\frac{C}{A}\frac{1}{1-\epsilon} - 1\right)\left(\frac{C}{A} - 1\right)\left(\frac{\omega_s}{p}\right)^2 \\ q &= \frac{1}{2}\frac{1-\epsilon}{1+\epsilon}\left[\frac{C}{2A}\frac{2-\epsilon}{1-\epsilon} - 1\right]\frac{\theta_0\omega_s}{p} - \frac{\eta}{4} \end{aligned}\right\} \quad (28)$$

By ignoring terms of the order η^2, we can reduce (24) into

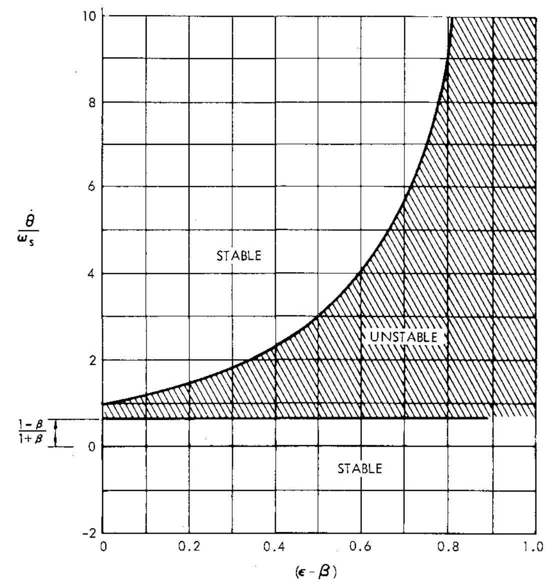

Fig. 5 Instability under circumferential motion of crew in noncentral-mass plane.

the standard form of Mathieu's equation

$$(d^2u/d\tau^2) + (a + 16q \cos 2\tau)u = 0 \quad (29)$$

The stability of the solution of this equation is well known.[1] Figure 6 shows the stability boundaries. If the physical parameters a and q are such that they fall in the shaded area, then it is possible to obtain a solution $u(\tau)$ in the form $e^{\mu\tau}\phi(\tau)$ with $\mu > 0$, where $\phi(\tau)$ is a periodic function of τ with period π or 2π. Then the amplitude of $u(\tau)$ grows exponentially as time increases. From Eq. (26) we see that this also implies an unbounded growth of the amplitude of ω_y'. Thus, Fig. 6 also determines the stability of ω_y' and, hence, of the space station.

Note that the constant a in (28) is proportional to the ratio $(\omega_s/p)^2$. Thus, stability of the space station depends on the ratio of the period of the astronaut's motion to that of the spin. If $\epsilon \ll 1$ and $C = 2A$, i.e., for a large satellite that is a circular disk, we have

$$a \doteq 4\left(\frac{\omega_s}{p}\right)^2 = \left(\frac{\text{period of crew motion}}{\text{half-period of satellite spin}}\right)^2 \qquad q \doteq [1 - (p^2/4\omega_s^2)](\theta_0\omega_s/2p) \quad (30)$$

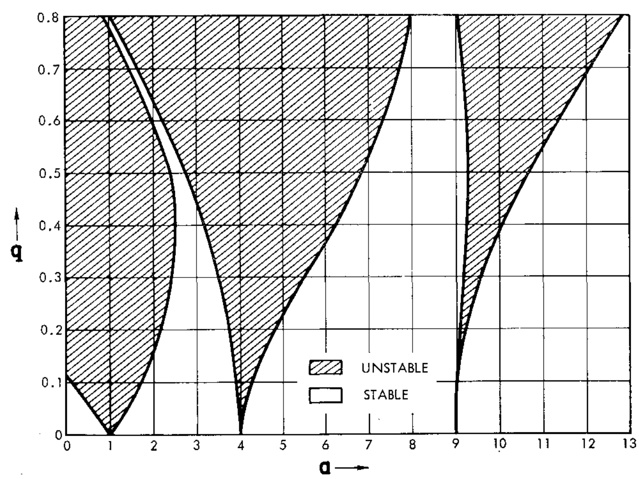

Fig. 6 Stability chart for crew undergoing harmonic oscillations. (For circumferential oscillations, see Eqs. (28) and (30) for q; for radial motion of crew, see Eq. (45) for q.)

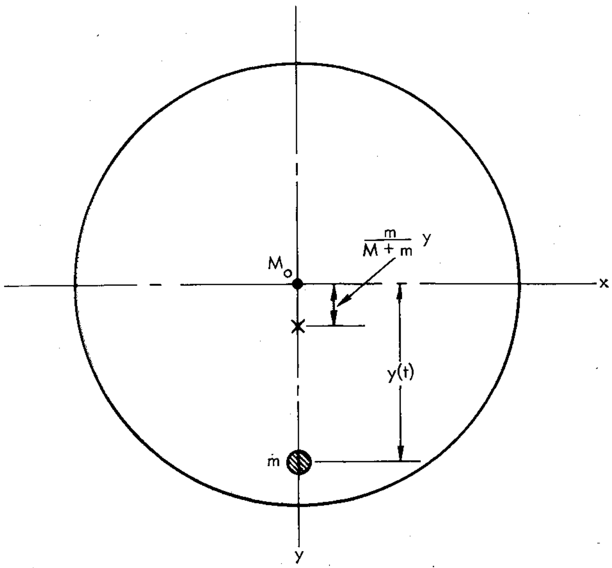

Fig. 7 Coordinates for radial motion of crew.

If θ_0 or q is very small, instability occurs only when $a \sim n^2$ ($n = 1, 2, 3, \ldots$). This means that the crew must not move periodically with a period that is an integral multiple of the half-period of the satellite's spin time. If the amplitude θ_0 or q is larger, the unstable region is enlarged. The astronaut's periodic motion must be even more restricted. Similar interpretation can be easily made for other ratios of C/A and ϵ.

Radial Motion of Crew

Consider a spinning, circular symmetric satellite of mass M, on which a smaller body of mass m is attached (Fig. 7). Let m oscillate radially in the plane of the body M. When m is at y, the centroidal moment of inertia matrix is

$$[I] = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}$$

where

$$\begin{aligned} I_x &= A + [mM/(M+m)]y^2(t) \\ I_y &= A \qquad I_z = C + [mM/(M+m)]y^2(t) \end{aligned} \tag{31}$$

and where A, A, and C are, respectively, the moments of inertia of the body M with respect to the x, y, and z axes passing through the geometric center. The matrix equation (7) can be reduced easily to the form

$$\begin{Bmatrix} \dot{\omega}_x' \\ \dot{\omega}_y' \\ \dot{\omega}_z' \end{Bmatrix} + \begin{bmatrix} \dot{I}_x/I_x & \omega_s & 0 \\ -\omega_s & 0 & 0 \\ 0 & 0 & \dot{I}_z/I_z \end{bmatrix} \begin{Bmatrix} \omega_x' \\ \omega_y' \\ \omega_z' \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -(\dot{I}_z/I_z)\omega_s \end{Bmatrix} \tag{32}$$

In this case, we choose Ω to be ω_s, the spin rate of the satellite.

The third equation in (32) is decoupled from the first two, as

$$\dot{\omega}_z' + (\dot{I}_z/I_z)\omega_z' = -(\dot{I}_z/I_z)\omega_s \tag{33}$$

which is integrated to be

$$\omega_z'(t) = [\alpha/I_z(t)] - \omega_s \tag{34}$$

where α is an integration constant. If the initial condition is $\omega_z'(0) = 0$, then

$$\omega_z'(t) = -\omega_s\{1 - [I_z(0)/I_z(t)]\} \tag{35}$$

Thus ω_z' is determined by the instantaneous moment of inertia.

The first two equations in (32) may be written, on dropping the primes over ω_x, ω_y, as

$$\begin{aligned} \dot{\omega}_x + (\dot{I}_x/I_x)\omega_x + \Omega\omega_y &= 0 \\ \dot{\omega}_y - \Omega\omega_x &= 0 \end{aligned} \tag{36}$$

Elimination of ω_y gives

$$\ddot{\omega}_x + (\dot{I}_x/I_x)\dot{\omega}_x + [(\dot{I}_x/I_x)^{\cdot} + \Omega^2]\omega_x = 0 \tag{37}$$

Here, I_x is a function of time

$$I_x(t) = A + [mM/(M+m)]y^2(t) \tag{38}$$

If $y(t)$ is a periodic function of time, then (37) is Hill's equation, the stability characteristics of which again have been investigated extensively in the literature (see Refs. 1–3).

It is a fact of considerable importance that the average value of the coefficient $\dot{\omega}_x'$ vanishes over a period; for, if $I_x(t+T) = I_x(t)$, then

$$\int_0^T \frac{\dot{I}_x}{I_x}\,dt = \log I_x\Big|_0^T = 0 \tag{39}$$

To reduce Eq. (37) to the standard form, let

$$\omega_x = \exp\left[-\int_0^t \frac{\dot{I}_x}{2I_x}\,dt\right]u = [I_x(t)]^{-1/2}u \tag{40}$$

Then Eq. (37) becomes

$$\frac{d^2u}{dt^2} + \left[\Omega^2 + \frac{d}{dt}\left(\frac{\dot{I}_x}{2I_x}\right) - \left(\frac{\dot{I}_x}{2I_x}\right)^2\right]u = 0 \tag{41}$$

By Floquet's theory,[1] the general solution of (41) is given by

$$u = \alpha_1 e^{\mu t}\phi(t) + \alpha_2 e^{-\mu t}\phi(t) \tag{42}$$

where α_1 and α_2 are arbitrary constants, μ is the characteristic exponent, and $\phi(t)$ is a periodic function in t whose period is the same as or twice the period of $I_x(t)$. From (42) and (40) it is readily seen, by virtue of (39), that the system is unstable whenever the real part of $\mu \neq 0$; for then u and ω_x and ω_y will be unbounded as $t \to \infty$.

Let $(\pi/T)t = \tau$, and write for the last terms in Eq. (41)

$$\frac{d}{d\tau}\left(\frac{1}{2I_x}\frac{dI_x}{d\tau}\right) - \left(\frac{1}{2I_x}\frac{dI_x}{d\tau}\right)^2 = \theta_0 + 2\sum_{\nu=1}^{\infty}(\theta_{\nu c}\cos 2\nu\tau + \theta_{\nu s}\sin 2\nu\tau)$$

provided that the series

$$\sum_{\nu=1}^{\infty}(|\theta_{\nu c}| + |\theta_{\nu s}|)$$

converges. Then Eq. (41) appears as

$$\frac{d^2u}{d\tau^2} + \left[\frac{T^2}{\pi^2}\Omega^2 + \theta_0 + 2\sum_{\nu=1}^{\infty}(\theta_{\nu c}\cos 2\nu\tau + \theta_{\nu s}\sin 2\nu\tau)\right]u = 0 \tag{43}$$

If only the coefficient θ_{1c} in the series on the right-hand side of Eq. (43) is different from zero, then we obtain the standard Mathieu's equation

$$(d^2u/d\tau^2) + [a + 16q\cos 2\tau]u = 0 \tag{44}$$

where

$$\left.\begin{aligned} a &= \left(\frac{T}{\pi}\Omega\right)^2 = \left(\frac{\text{period of crew motion}}{\text{half-period of satellite spin}}\right)^2 \\ q &= \frac{1}{8}\theta_{1c} = \frac{1}{8\pi}\int_0^\pi \left[\frac{d}{d\tau}\left(\frac{1}{2I_x}\frac{dI_x}{d\tau}\right) - \left(\frac{1}{2I_x}\frac{dI_x}{d\tau}\right)^2\right]\cos\tau\,d\tau \end{aligned}\right\} \tag{45}$$

The stability diagram of Eq. (44) is shown in Fig. 6. The larger the mass of the astronaut and the farther he moves, the larger will be the value of q. When q is very small, instability may occur if the man moves periodically along a radius with a period that is an integral multiple of the half-period of the spin of the satellite. If the man's weight is larger, or if his amplitude of motion increases, the unstable regime will be wider.

If the other coefficients $\theta_0, \theta_{1s}, \theta_{2c}, \theta_{2s}, \ldots$ in Eq. (43) cannot be ignored, we must deal with Hill's equation (43). The stability diagram is fairly similar to Fig. 6.

Conclusions

We have shown that, if a man on a spinning space station moves periodically, he can rock the station and cause it to tumble. The crucial parameter, of course, is the period of his motion. Usually he should avoid having a periodic motion with a period in the neighborhood of an integral multiple of the half-period of the spin of the satellite. The exact forbidden intervals of the periods of motion depend on the type and amplitude of his motion, the satellite's geometry and inertia, the man's mass, etc. Several examples are given in this paper.

References

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Tumbling Motions of an Artificial Satellite

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This paper treats the motion of a passively damped, gravity stabilized, artificial satellite that is tumbling or rotating about an arbitrary axis. This type of motion may occur after launch and separation, but before "capture" into a libration motion. A perturbation method is applied to this nonlinear problem. It is shown that the gravity gradient torques cause internal friction with consequent decrease of the body angular rates. Results are obtained for decay of the tumbling rate from an initial rate to the beginning of bounded libration. It is found that, for tumbling angular rates greater than three times the mean orbit angular rate, the time to capture increases as the cube of the initial rate.

Nomenclature

$(d/d\tau)(\ )$	$= d/d(nt)(\ ) = (\ )'$
n	= orbital angular rate
τ	$= nt$ = normalized time
ϕ, θ, γ	= Euler angles of the body axes
$\hat{1}, \hat{2}, \hat{3}$	= orbital reference axes unit vectors
$\hat{1}b, \hat{2}b, \hat{3}b$	= body principal axes unit vectors
I_1, I_2, I_3	= body moments of inertia
J_{11}, J_{12}, J_{13}	= wheel number one moments of inertia about the $\hat{1}b, \hat{2}b, \hat{3}b$ axes, respectively
J_{31}, J_{32}, J_{33}	= wheel number three moments of inertia about $\hat{1}b, \hat{2}b, \hat{3}b$ axes, respectively
ψ_1, ψ_3	= wheel rotation angles
$\omega_1, \omega_2, \omega_3$	= body angular rates with respect to inertial space about $\hat{1}b, \hat{2}b, \hat{3}b$ axes
$\hat{R}^c$	= unit vector from satellite center of mass to the center of attraction
I_1'	$= I_1 + J_{31}$
I_2'	$= I_2 + J_{12} + J_{32}$
I_3'	$= I_3 + J_{13}$
k_1	$= (I_3' - I_2' + J_{33})/I_1'$ = body shape parameter
k_2	$= (I_3' - I_1' + J_{33} - J_{11})/I_2'$ = body shape parameter
k_3	$= (I_2' - I_1' - J_{11})/I_3'$ = body shape parameter
K_1, K_3	= spring resonant frequencies
η_1, η_2	= damping constants
r_1	$= J_{11}/I_1'$
r_3	$= J_{33}/I_3'$
ν_1	$= J_{33}/I_1'$
ν_2	$= J_{11}/I_2'$
ν_3	$= J_{33}/I_2'$
ν_4	$= J_{11}/I_3'$
g_1	$= -(k_1/2)\sin^2\phi \sin 2\theta$
g_2	$= (k_2/2)\sin\theta \sin 2\phi$
g_3	$= (k_3/2)\cos\theta \sin 2\phi$
h_1	$= -(k_1/2)\sin\theta \sin 2\phi$
h_2	$= -(k_2/2)\sin^2\theta \sin 2\phi$
h_3	$= -(k_3/2)[\sin^2\phi \cos^2\theta - \cos^2\phi]$
γ	= body pitch angle
λ	= parameter (greater than one) used in perturbation series
ω	$= \gamma'/\lambda > 1$
η	$= (1 + r_3)\eta_3$ = damping parameter
ω_0	$= (1 + r_3)^{1/2}K_3$ = natural frequency
ψ	$= \psi_3$ = wheel angle of rotation
k	$= \frac{3}{2}k_3$ = body shape parameter
r	$= r_3/(1 + r_3)$ = inertia ratio
Ω	$= \psi'$ = wheel angular rate
H	= Hamiltonian for pitch motion
H_0	= initial value of H
γ_c, γ_c'	= variables evaluated at τ_c, the instant of capture
$\bar{\Omega}, \bar{\omega}, \bar{\psi}, \bar{\gamma}$	= long period or "average" value of $\Omega, \omega, \psi, \gamma$
$\tilde{\Omega}, \tilde{\omega}, \tilde{\psi}, \tilde{\gamma}$	= refinement in approximation of $\Omega, \omega, \psi, \gamma$
α, β, D	= defined in Eq. (7)
N	$= \lambda\bar{\omega}$ = average pitch tumbling rate
N_0	= initial value of N
N_c	= value of N at capture
$(\tau_c)_{\min}$	= minimum time to go from $\tau = 0$ to $\tau = \tau_c$ with a given N_0
$\eta_{\text{opt}}, (\omega_0^2)_{\text{opt}}$	= optimum values of η, ω_0^2 giving $\tau = (\tau_c)_{\min}$

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